

LOCAL INDICATORS FOR PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. - The notion of index, classical in number theory and extended in [18] to plurisubharmonic functions, allows to define an indicator which is applied to the study of the Monge-Ampère operator and a pluricomplex Green function.

1 Introduction

We recall some local notions which are often used in various investigations.

a). The index $I(F, x^0, a)$ of a zero $x^0 \in \Omega$ of a holomorphic function $F \in Hol(\Omega)$, where Ω is a domain of \mathbf{C}^n , is used in important results of number theory [12]; $I(F, x^0, a)$ is defined by means of the set $\omega \in \mathbf{N}^n$ of n -tuples $(i) = (i_1, \dots, i_n)$ such that $D^{(i)}F(x^0) \neq 0$. Given a direction $(a) = (a_k > 0; 1 \leq k \leq n) \in \mathbf{R}^n$,

$$I(F, x^0, a) = \inf_{(i)} (a, i) \quad \text{for } (a, i) = \sum_k a_k i_k \geq 0 \text{ and } (i) \in \omega. \quad (1)$$

b). In fact, the index $I(F, x^0, a)$ is a property (function of a and x^0) of the current of integration $[W] = dd^c \log |F|$ over the analytic set $W = \{x \in \Omega : F(x) = 0\}$, where $d = \partial + \bar{\partial}$ and $d^c = (2\pi i)^{-1}(\partial - \bar{\partial})$. We denote by $PSH(\Omega)$ the class of plurisubharmonic functions in a domain Ω of \mathbf{C}^n and by $\Theta_p(\Omega)$ the class of positive closed currents represented by homogeneous forms of $dx_k, d\bar{x}_k$, $1 \leq k \leq n$, of bidegree $(n-p, n-p)$. The Lelong number $\nu(T, x^0)$ at $x^0 \in \Omega$ for $T \in \Theta_p(\Omega)$ is related to the trace measure of T ,

$$\sigma_T = T \wedge \beta_p, \quad (2)$$

where $\beta_p = (p!)^{-1} \beta_1^p$ is the volume element of \mathbf{C}^p . By the definition,

$$\nu(T, x^0) = \lim_{r \rightarrow 0} (\tau_{2p} r^{2p})^{-1} \sigma_T[B^{2n}(x^0, r)] \quad (3)$$

where τ_{2p} is the volume of the unit ball $B^{2p}(0, 1)$ of \mathbf{C}^p . In (3), the trace measure σ_T belongs to the remarkable class $\mathcal{P}_p(\Omega)$ of positive measures characterized by the property that the quotient in the right hand side of (3) is an increasing function of

r (see [15]). Another definition of the number $\nu(T, x^0)$ is derived from (2) and (3) by setting $\varphi_1(x) = \log |x - x^0|$:

$$\nu(T, x^0) = \lim_{t \rightarrow -\infty} \int_{\{\varphi_1 < t\}} T \wedge (dd^c \varphi_1)^p. \quad (4)$$

c). By replacing in (4) φ_1 with a function $\varphi \in PSH(\Omega)$ such that $\exp \varphi$ is continuous and the set $\{\varphi(x) = -\infty\}$ is relatively compact, J.-P. Demailly (see [4] and [5]) has found new important applications of (4). The choice of the function

$$\varphi_a(x) = \sup_k a_k^{-1} \log |x_k - x_k^0| \quad (5)$$

which is circled with the center x^0 and such that $\{\varphi_a(x) = -\infty\}$ is reduced to x^0 , allows us to put into this framework the notion of index $I(F, x^0, a)$ of a zero of a holomorphic function F at x^0 . As was proved in [18],

$$I(F, x^0, a) = (a_1 \cdot a_2 \dots a_n) \nu[T, x^0, \varphi_a(x - x^0)] \quad (6)$$

for $T = dd^c \log |F| \in \Theta_1(\Omega)$. It can be extended to arbitrary plurisubharmonic functions by setting for $f \in PSH(\Omega)$ and $x^0 \in \Omega$,

$$n(f, x^0, a) = (a_1 \cdot a_2 \dots a_n) \nu[dd^c f, x^0, \varphi_a(x - x^0)], \quad (7)$$

so that

$$I(F, x^0, a) = n(f, x^0, a) \quad \text{for } f = \log |F|. \quad (8)$$

d). In the recent paper [18], $n(f, x^0, a)$ has appeared in a simpler form given for $x^0 = 0$ by the relation

$$n(f, 0, a) = \lim_{w \rightarrow 0} (\log w)^{-1} f(w, x, a), \quad 0 \leq w \leq 1, \quad x \in \Omega \setminus A, \quad (9)$$

where $f(w, x, a) = f(w^{a_1} x_1, \dots, w^{a_n} x_n)$ and A is an algebraic set for $f = \log |F|$. In the general case, for $f \in PSH(\Omega)$, A is of zero measure and

$$n(f, 0, a) = \liminf_{w \rightarrow 0} (\log w)^{-1} f(w, x, a)$$

outside a set $A' \subset A$ which is pluripolar in Ω .

In the first part of the present paper we will use (9) for a study of singularities of plurisubharmonic functions f , supposing $f \in PSH(\Omega)$ and $D \subset \subset \Omega$, where D is the unit polydisk $\{x \in \mathbf{C}^n : \sup |x_k| < 1\}$; we denote by $PSH_-(D)$ the class of f satisfying $\sup \{f(x) : x \in D\} \leq 0$ and $f \not\equiv -\infty$. The domain D as well as the weights φ_a in (5) being circled, it is natural to work with a circled image f_c of f and then with its convex image on the space \mathbf{R}_-^n of $u_k = \log |x_k|$, $1 \leq k \leq n$.

Developing the results of J.-P. Demailly and C.O. Kiselman [10] we get the value $n(f, 0, a)$ which produces, via $a_k = -\log |y_k|$, a function $\Psi_{f,0}(y)$, the local indicator of f at 0, which is plurisubharmonic in $D_{(y)}$, the unit polydisk in the space \mathbf{C}_y^n . It satisfies the Monge-Ampère equation

$$(dd^c \Psi_{f,0})^n = \tau_f(0) \delta(0), \quad \tau_f(0) > 0, \quad (10)$$

where $\delta(0)$ is the Dirac measure at the origin of \mathbf{C}_y^n , and

$$(dd^c f)^n \geq \tau_f(x^0) \delta(x^0) \quad (11)$$

for $f \in PSH(\Omega)$ such that $(dd^c f)^n$ is well defined, and $x^0 \in \Omega$. Moreover, there is the relation $\tau_f(x^0) \geq [\nu(T, x^0)]^n$ for $T = dd^c f \in \Theta_{n-1}(\Omega)$.

Then we consider a class of plurisubharmonic functions on Ω with singularities on a finite set $\{x^1, \dots, x^N\}$, controlled by given indicators Ψ_m , $1 \leq m \leq N$. We construct a plurisubharmonic function G vanishing on $\partial\Omega$ and such that $\Psi_{G,x^m} = \Psi_m$, $1 \leq m \leq N$, and $(dd^c G)^n = \sum_m \tau_m \delta(x^m)$ with τ_m the mass of $(dd^c \Psi_m)^n$. We prove that G is the unique plurisubharmonic function with these properties. For the case $\Psi_m(x) = \nu_m \log |x - x^m|$, it coincides with the pluricomplex Green function with weighted poles at x^1, \dots, x^N ([16]). We prove a variant of comparison theorem for plurisubharmonic functions with controlled singularities and study a Dirichlet problem for this class of functions.

A part of the results of Section 3 are close to those from [19] where a weighted pluricomplex Green function with infinite singular set was introduced and the corresponding Dirichlet problem was studied.

2 Circled functions and convex projections

We will consider here the case $x^0 = 0$ and $f \in PSH(\Omega)$ supposing $0 \in D \subset \subset \Omega$, where D is the unit polydisk $\{x \in \mathbf{C}^n : \sup |x_k| < 1\}$, and $f \in PSH_-(D)$, that is $f(x) \leq 0$ for $x \in \overline{D}$ and $f \not\equiv -\infty$.

A set $A \subset \mathbf{C}^n$ is called *0-circled* (or just *circled*) if $x = (x_k) \in A$ implies $x' = (x_k e^{i\theta_k}) \in A$ for $0 \leq \theta_k \leq 2\pi$, $1 \leq k \leq n$. We will say that a function $f(x)$ defined on A , is circled if it is invariant with respect to the rotations $x_k \mapsto x_k e^{i\theta_k}$, $1 \leq k \leq n$.

Given a function $f \in PSH(\Omega)$, Ω being a circled domain, we consider a circled function $f_c \in PSH(\Omega)$ equal to the mean value of $f(x_k e^{i\theta_k})$ with respect to $0 \leq \theta_k \leq 2\pi$, $1 \leq k \leq n$. In what follows, we will also use another circled function $f'_c \geq f_c$, equal to the maximum of $f(x_k e^{i\theta_k})$ for $0 \leq \theta_k \leq 2\pi$, $1 \leq k \leq n$. Note that the differential operators, namely ∂ , $\bar{\partial}$, $d = \partial + \bar{\partial}$ and $d^c = (2\pi i)^{-1}(\partial - \bar{\partial})$, commute with the mapping $f \mapsto f_c$, so $(\partial f)_c = \partial f_c$, however it is not the case for $f \mapsto f'_c$.

To a Radon measure σ on a circled domain Ω , we relate a circled measure σ_c defined by $\sigma_c(f) = \sigma(f_c)$ for continuous functions f . In the same way, to a current $T \in \Theta_p(\Omega)$ we associate a circled current T_c which is defined on homogeneous

forms λ of bidegree (p, p) by $T_c(\lambda) = T(\lambda_c)$, where λ_c are obtained by replacing the coefficients of λ with their mean values with respect to θ_k . In particular, if $T = dd^c f$, $f \in PSH(\Omega)$ and Ω circled, $T_c = dd^c f_c$. It gives us a specific property of the value $n(f, 0, a)$ defined by (4) and (6), and of the index $I(F, 0, a)$.

Proposition 1 *Let Ω be a 0-circled domain and $T \in \Theta_p(\Omega)$. For every 0-circled weight φ , $\nu(T, \varphi) = \nu(T_c, \varphi)$. In particular, since the weight φ_a defined by (5) for $x^0 = 0$, is circled, the number $n(f, 0, a)$ in (7) for $f \in PSH(\Omega)$ at the origin can be calculated by replacing f with f_c : $n(f, 0, a) = n(f_c, 0, a)$, and the number $n(f, 0, a)$ is calculated on the convex image $g(u)$ of f_c , $g(u) = f[\exp(u_k + i\theta_k)]$:*

$$n(f, 0, a) = \lim_{v \rightarrow -\infty} v^{-1} g(u_k + a_k v).$$

For $f \in PSH_-(D)$, as was shown in [18],

$$n(f, 0, a) = \lim_{w \rightarrow 0} (\log w)^{-1} f(w^{a_1} x_1, \dots, w^{a_n} x_n)$$

for almost all $x \in D$. The limit exists for all $x \in D$ when replacing $f(x)$ by $f_c(x)$ or by $f'_c(x)$. Indeed, for $R > 1$

$$f_c(x) \leq f'_c(x) \leq \gamma_R f_c(Rx) \leq 0 \quad (12)$$

with $\gamma_R = (R - 1)^n (R + 1)^{-n}$ which satisfies $1 - \epsilon \leq \gamma_R \leq 1$ for $R > R_0(\epsilon)$.

The calculation of $n(f, 0, a)$ for $f \in PSH_-(D)$ uses the convex image $g_f(u) = f_c[\exp(u_k + i\theta_k)]$ or $g'_f(u) = f'_c[\exp(u_k + i\theta_k)]$ obtained by setting $x_k = \exp(u_k + i\theta_k)$, the functions g_f and g'_f being defined on $\mathbf{R}_-^n = \{-\infty \leq u_k \leq 0\}$.

Proposition 2 *In order that a function $h(u_1, \dots, u_n) : \mathbf{R}_-^n \rightarrow \mathbf{R}_-$ be the image of $f \in PSH_-(D)$ obtained by $x_k = \exp(u_k + i\theta_k)$ and*

$$f(x) = f[\exp(u_k + i\theta_k)] = h(u),$$

it is necessary and sufficient that h be convex of $u \in \mathbf{R}_-^n$, increasing in each u_k , $-\infty \leq u_k \leq 0$, and $h(u) \not\equiv -\infty$, $h(u) \leq 0$.

The necessity condition results from the classic properties of $f \in PSH_-(D)$. To show the sufficiency, we remark that convexity of h implies its continuity on \mathbf{R}_-^n . On the other hand, we have (the derivatives being taken in the sense of distributions), for any $\lambda \in \mathbf{C}^n$,

$$4 \sum \frac{\partial^2}{\partial x_k \partial \bar{x}_j} \lambda_k \bar{\lambda}_j = \sum \frac{\partial^2}{\partial u_k \partial u_j} \lambda'_k \bar{\lambda}'_j \quad (13)$$

where $\lambda'_k = x_k^{-1} \lambda_k$. Let $A \subset D$ be the union of the subspaces $\{x_k = 0\}$ in D . By (13), $f \in PSH(D \setminus A)$. The condition $f(x) \leq 0$ implies that f extends by upper semicontinuity to A , so $f \in PSH_-(D)$ for $f(x) = h(\log |x_1|, \dots, \log |x_n|)$.

Definition. We denote by $Conv(\mathbf{R}_-^n)$ the class of functions $h(u_1, \dots, u_n) \leq 0$, $-\infty \leq u_k \leq 0$, satisfying the conditions listed in Proposition 2.

Proposition 3 *Let $h \in Conv(\mathbf{R}_-^n)$ be the image of $f(x_k) = h[\exp(u_k + i\theta_k)] \in PSH_-(D)$. Then*

a)

$$\lim_{v \rightarrow -\infty} v^{-1} h(u_1 + v, \dots, u_n + v) = \lim_{v \rightarrow -\infty} \frac{\partial}{\partial v} h(u_1 + v, \dots, u_n + v) = \nu(f, 0),$$

where $\nu(f, 0)$ is the Lelong number of f at $x = 0$. More generally,

$$\lim_{v \rightarrow -\infty} v^{-1} h(u_1 + a_1 v, \dots, u_n + a_n v) = n(f, 0, a) \quad (14)$$

is independent of u_k ;

b) $\lim_{v \rightarrow -\infty} v^{-1} h(u_1 + v, u_2, \dots, u_n) = \nu_1(f, 0)$ is independent of u_k and is the generic Lelong number (cf. [7]) of the current $T = dd^c f$ along the variety $D_1 = \{x \in D : x_1 = 0\}$. Moreover, for $x'_1 = (x_2, \dots, x_n)$, the function

$$h_1(r_1, x'_1) = (2\pi)^{-1} \int_0^{2\pi} f(r_1 e^{i\theta_1}, x'_1) d\theta_1,$$

has the property

$$\lim_{w \rightarrow 0} (\log w)^{-1} h_1(wr_1, x'_1) = \nu_1(f, 0)$$

for $x'_1 \in D_1$ with exception of a pluripolar subset of D_1 ;

c) $\sum_1^n \nu_k(f, 0) \leq \nu(f, 0)$.

Proof. Existence and equality of the limits in a) follow from the increasing with respect to v , $-\infty < v \leq 0$, and from the condition $h \leq 0$. Moreover, if $l(\rho)$ is the mean value of $f(x)$ over the sphere $|x| = \rho$, then

$$\nu(f, 0) = \lim_{\rho \rightarrow 0} \frac{\partial l(\rho)}{\partial \log \rho} = \lim_{\rho \rightarrow 0} (\log \rho)^{-1} l(\rho). \quad (15)$$

We compare the mean values with respect to θ_k over the circled domains $B(0, \rho)$ and $D(\rho) = \{\sup |x_k| \leq \rho < 1\}$ for the image $h(u)$ of the circled function $f_c(x)$, for $u_k = \log \rho - \frac{1}{2} \log n$ and $u'_k = \log \rho$, $1 \leq k \leq n$:

$$h(u) \leq l(\log \rho) \leq h(u'),$$

since $D(\rho/\sqrt{n}) \subset B(0, \rho) \subset D(\rho)$. This gives us (14) and a).

Statement b) is known (cf. [15]). The limit

$$-c(x'_1) = \lim_{r \rightarrow 0} \left(\log \frac{1}{r} \right)^{-1} h(r, x'_1) \leq 0 \quad (16)$$

for $r \searrow 0$ exists and is obtained by increasing negative values, the second term of (16) belonging to $PSH(D_1)$ for $r > 0$. If $c(\hat{x}'_1) = 0$ for a point $\hat{x}'_1 \in D_1$, then $c(x'_1) = 0$ except for a pluripolar subset of D and the statement is proved. Otherwise, consider the set $\overline{D}_1(r) \subset\subset D_1$ and $c_0 = \sup c(x'_1)$ for $x'_1 \in D_1(r)$ and apply the preceding argument to

$$\lim_{r \rightarrow 0} \left[\left(\log \frac{1}{r} \right)^{-1} h(r, x'_1) + c_0 \right].$$

The statement for $h \in \text{Conv}(\mathbf{R}^n_-)$ follows from this precise property of the plurisubharmonic image.

To establish c), we observe that for $u \in \mathbf{R}^n_-$ and $h(u)$ the image in $\text{Conv}(\mathbf{R}^n_-)$ of $f \in PSH_-(\overline{D})$,

$$\frac{\partial}{\partial v} h(u_1 + v, \dots, u_n + v) = \sum_1^n \frac{\partial h}{\partial u_k}(u_1 + v, \dots, u_n + v),$$

the derivatives are positive and decreasing for $v \searrow -\infty$, and the limit of

$$\frac{\partial h}{\partial u_k}(u_1 + v, u_2, \dots, u_n)$$

is equal to $\nu_1(f, 0)$, the Lelong number of $dd^c f$ along D_1 . Therefore

$$\frac{\partial}{\partial v} h(u_1 + v, \dots, u_n + v) \geq \sum_1^n \nu_k(f, 0),$$

so taking $v \searrow -\infty$ we get by a),

$$\nu(f, 0) \geq \sum_1^n \nu_k(f, 0). \quad (17)$$

Remark. Actually, by the theorem of Y.T. Siu, (17) is a particular case of the following statement: the number $\nu(f, 0)$ is at least equal to the sum of the generic numbers $\nu(W_i)$ for $T = dd^c f$ along analytic varieties W_j of codimension 1 containing the origin.

In what follows, we will use a special subclass of circled plurisubharmonic functions $f \in PSH_-(D)$ that have the following "conic" property: the convex image $g_f(u)$ of f satisfies the equation

$$g_f(cu) = c g_f(u) \text{ for every } c > 0. \quad (18)$$

Such a function f will be called an *indicator*. For example, the weights φ_a in (5) are indicators.

Proposition 4 *Let $f \in PSH_-(D)$ be an indicator. Then $(dd^c f)^n = 0$ on $D_0 = \{x \in D : x_1 \dots x_n \neq 0\}$.*

Proof. It is sufficient to show that the domain D_0 can be foliated by one-dimensional analytic varieties γ_y such that the restriction of f to each leaf γ_y is harmonic on γ_y . So, given $y = (|y_k|e^{i\theta_k}) \in D_0$, consider an analytic variety γ_y , the image of \mathbf{C} under the holomorphic mapping $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_k(\zeta) = |y_k|^\zeta e^{i\theta_k}$. Note that $y = \lambda(1) \in \gamma_y$. As f is circled, the function $f_y(\zeta) = f(\lambda(\zeta))$, the restriction of f to γ_y , depends only on $\operatorname{Re} \zeta$. By (18), $f_y(\zeta)$ satisfies $f_y(c\zeta) = c f_y(\zeta)$ for all $c > 0$. Therefore, it is linear and thus harmonic on γ_y .

3 Indicator of a plurisubharmonic function

Given a function $f \in PSH(\Omega)$ and a point $x^0 \in \Omega$, we will construct a function $\Psi_{f,x^0}(y)$ related to local properties of f at x^0 . We will have $\Psi_{f,x^0} \in PSH_-(D)$, D being the unit polydisk in the space $\mathbf{C}_{(y)}^n$, and $\Psi_{f,x^0}(y) < 0$ in D if and only if the Lelong number of f at x^0 is strictly positive, otherwise $\Psi_{f,x^0}(y) \equiv 0$.

Definition. The *local indicator* (or just *indicator*) $\Psi_{f,0}$ of a function $f \in PSH_-(D)$, $D \subset \mathbf{C}_{(x)}^n$, at $x^0 = 0$ is defined for $y \in D \subset \mathbf{C}_{(y)}^n$ by

$$\Psi_{f,0}(y) = -n(f, 0, -\log |y_k|).$$

Referring to (9) with $R = -\log w$, $0 < R < +\infty$, we rewrite this as

$$\Psi_{f,0}(y) = \lim_{R \rightarrow +\infty} R^{-1} f[\exp(u_k + i\theta_k + R \log |y_k|)]. \quad (19)$$

The limit (19) exists *almost everywhere* for $x_k = \exp(u_k + i\theta_k)$, however (see Introduction) the value $n(f, 0, a)$ can be calculated as well by replacing $f(x)$ with the circled functions $f_c(x)$ or $f'_c(x)$. One can then substitute them for f in (19) to get $\Psi_{f,0}$. At $x^0 \neq 0$, the function $\Psi_{f,x^0}(y)$ is defined by means of $f[x_k^0 + \exp(u_k + i\theta_k + R \log |y_k|)]$.

If f is replaced by $f_c[\exp(u_k + i\theta_k)] = g_f(u_k)$ or by $f'_c[\exp(u_k + i\theta_k)] = g'_f(u_k)$, the limit exists, by Proposition 3, for *every* $u = (u_k) \in \mathbf{R}_-^n$:

$$\Psi_{f,0}(y) = \lim_{R \rightarrow +\infty} R^{-1} g(u_k + R \log |y_k|), \quad (20)$$

and does not depend on u .

Proposition 5 *Let $f \in PSH_-(D)$. Then*

a) $\Psi_{f,0}(y) \in PSH_-(D)$ and is 0-circled;

b) the convex image $g_\psi(u)$ in \mathbf{R}_-^n has the conic property $g_\psi(cu) = cg_\psi(u)$ for every $c > 0$, i.e. $\Psi_{f,0}$ is an indicator;

c)

$$\Psi_{f,0}(y) \geq f'_c(y) \geq f(y), \quad \forall y \in D; \quad (21)$$

d) the mapping $f \mapsto \Psi_{f,0}$ is a projection, $\Psi_{f,0}(y)$ is its own indicator at the origin;

e) the indicator $\Psi_{f,0}$ is the least indicator majorizing f on D ;

f) if $f_j(x_j)$ is the restriction of f to the complex subspace $\{x_s = 0, \forall s \neq j\}$ and

$$f_j(x_j) \not\equiv -\infty, \quad (22)$$

then $\Psi_{f,0}(y) \geq \nu_j \log |y_j|$, ν_j being the Lelong number of $f_j(x_j)$ at the origin;

g) if (22) holds for each j , then the Monge-Ampère operator $(dd^c \Psi_{f,0})^n$ is well defined on the whole polydisk D and

$$(dd^c \Psi_{f,0})^n = 0 \quad (23)$$

on $D \setminus \{0\}$.

Proof. Statement a) follows from (20), $g(R \log |y_k|)$ being a convex negative function for $R > 0$, and the limit of the quotient is obtained by increasing negative values. When setting $v_k = \log |y_k| = -a_k$, the image of $\Psi_{f,0}$ belongs to $Conv(\mathbf{R}_-^n)$ and $\Psi_{f,0}(y)$ is a 0-circled plurisubharmonic function.

The property b), essential for the indicator $\Psi_{f,0}$, results from the equality $n(f, 0, ca) = cn(f, 0, a)$ for all $c > 0$.

Relations c) are a consequence of (20) where $g(u)$ is the convex image $g'_f(u)$ of $f'_c(x) = \sup_{\theta_k} f(x_k e^{i\theta_k})$. We have $g'_f(\log |y_k|) \geq f(y)$. On the other hand, the quotient $m(R) = R^{-1} g'_f(R \log |y_k|)$, $R > R_0 > 1$, is a convex, negative and increasing function of R for $|y_k| < 1$. Therefore, $\lim_{R \rightarrow +\infty} m(R) \geq m(1)$, and by (20),

$$0 \geq \Psi_{f,0}(y) \geq g'_f(\log |y_k|) \geq f(y)$$

for $y \in D$.

Statement d) follows from (20) for $f = \Psi_{f,0}$ and from relation b).

To prove e), consider any indicator $\psi(y) \geq f(y)$ on D . Then $\Psi_{\psi,0}(y) \geq \Psi_{f,0}(y)$, and by d), $\Psi_{\psi,0} = \psi$, so $\psi(y) \geq \Psi_{f,0}(y) \forall y \in D$.

The bound in f) results from c) and the maximum principle for plurisubharmonic functions, since (for $j = 1$)

$$\Psi_{f,0}(y) \geq \sup_{\theta_k} f(y_k e^{i\theta_k}) \geq \sup_{\theta_1} f(y_1 e^{i\theta_1}, 0, \dots, 0)$$

and for $|y_1| \searrow 0$ the quotient $(\log |y_1|)^{-1} \sup_{\theta_1} f_1(y_1 e^{i\theta_1})$ for the restriction f_1 to the complex subspace $\{x_s = 0, \forall s > 1\}$, decreases to ν_1 .

Finally, in the assumptions of g), the function $\Psi_{f,0}(y)$ is locally bounded on $D \setminus \{0\}$ by f), so the operator $(dd^c \Psi_{f,0})^n$ is well defined on D . Equation (23) is valid on the domain $D \setminus \{y : y_1 y_2 \dots y_n = 0\}$ by Proposition 4 and then on $D \setminus \{0\}$, because the Monge-Ampère measure of a bounded plurisubharmonic function has zero mass on any pluripolar set (see [2]).

Remark. Statement d) of Proposition 5 is, in other words, that all the directional numbers $\nu(dd^c \Psi_{f,0}, \varphi_a)$ of $\Psi_{f,0}$ coincide with the directional numbers $\nu(dd^c f, \varphi_a)$ of the original function f , $\forall a \in \mathbf{R}_+^n$.

The above construction is in fact of local character and Proposition 5 remains valid for the indicator Ψ_{f,x^0} of any function $f(x)$ plurisubharmonic in a neighbourhood ω of a point $x^0 \in \mathbf{C}^n$, with the the following change in the statement c): (21) should be replaced by

$$\Psi_{f,x^0}(x - x^0) \geq f(x) + C \quad \forall x \in D(x^0, r), \quad C = C(u, r), \quad (24)$$

where $D(x^0, r) = \{x : |x_k - x_k^0| < r, 1 \leq k \leq n\}$ and $r > 0$ is such that the polydisk $D(x^0, r) \subset\subset \omega$. And of course the restriction f_j in (22) should be taken to the subspaces $\{x_s = x_s^0, \forall s \neq j\}$.

Let now $f(x) \in PSH(\omega)$ be locally bounded on $\omega \setminus \{x^0\}$. Then its indicator Ψ_{f,x^0} satisfies the equation

$$(dd^c \Psi_{f,x^0})^n = \tau_f(x^0) \delta(0) \quad (25)$$

with some number $\tau_f(x^0) \geq 0$ and $\delta(0)$ the Dirac measure at 0, and $\tau_f(x^0) > 0$ if and only if the Lelong number of the function f at x^0 is strictly positive. And now we relate this value to $(dd^c f)^n$.

Theorem 1 *Let $f \in PSH(\omega)$ be locally bounded out of a point $0 \in \omega$. Then*

$$(dd^c f)^n \geq \tau_f(0) \delta(0). \quad (26)$$

Proof. In view of (24), the function f satisfies

$$\limsup_{x \rightarrow 0} \frac{\Psi_{f,0}(x)}{f(x)} \leq 1. \quad (27)$$

By the Comparison theorem of Demailly [7], Theorem 5.9, this implies

$$(dd^c \Psi_{f,0})^n|_{\{0\}} \leq (dd^c f)^n|_{\{0\}}.$$

On the other hand,

$$(dd^c \Psi_{f,0})^n|_{\{0\}} = (dd^c \Psi_{f,0})^n = \tau_f(0) \delta(0)$$

by (25), that gives us (26).

The theorem is proved.

Remark. It is well known that for any plurisubharmonic function v with isolated singularity at 0, there is the relation

$$(dd^c v)^n \geq [\nu(dd^c v, 0)]^n \delta(0). \quad (28)$$

By the remark after the proof of Proposition 5, $\nu(dd^c f, 0)$ is equal to $\nu(dd^c \Psi_{f,0}, 0)$. Applying (28) to $v = \Psi_{f,0}$ we get, in view of Theorem 1,

$$(dd^c f)^n \geq (dd^c \Psi_{f,0})^n \geq [\nu(dd^c \Psi_{f,0}, 0)]^n = [\nu(dd^c f, 0)]^n,$$

so (26) is an improvement of (28).

For example, if $f(x) = \log(|x_1|^{k_1} + |x_2|^{k_2})$ with $0 < k_1 < k_2$, then

$$(dd^c f)^2 = \tau_f(0) \delta(0) = k_1 k_2 \delta(0) > k_1^2 \delta(0) = [\nu(dd^c f, 0)]^2 \delta(0),$$

and thus $\tau_f(0) > [\nu(dd^c f, 0)]^2$.

More generally, if F is a holomorphic mapping to \mathbf{C}^n with an isolated zero at 0 of multiplicity μ_0 , and $f = \log |F|$, then

$$[\nu(dd^c f, 0)]^n \leq \tau_f(0) \leq \mu_0.$$

In fact, relation (27) makes it possible to obtain extra bounds for $(dd^c f)^n$ in case of $\exp f \in C(\Omega)$. Such a function f can be then considered as a plurisubharmonic weight φ for Demailly's generalized numbers $\nu(T, \varphi)$ of a closed positive current T of bidimension (p, p) , $1 \leq p \leq n - 1$ [7]:

$$\nu(T, \varphi) = \lim_{s \rightarrow -\infty} \int_{\{\varphi < s\}} T \wedge (dd^c \varphi)^p = T \wedge (dd^c \varphi)^p|_{\{0\}}.$$

Moreover, the function $\Psi_{f,0}$ is such a weight, too. By Comparison theorem from [7], Theorem 5.1, relation (27) implies

$$\nu(T, \Psi_{f,0}) \leq \nu(T, f). \quad (29)$$

Take

$$T_k = (dd^c f)^k \wedge (dd^c \Psi_{f,0})^{n-k-1}, \quad 1 \leq k \leq n-1.$$

These currents are well defined on a neighbourhood of 0 and are of bidimension $(1, 1)$. Applying (29) to $T = T_k$ we obtain

$$T_k \wedge dd^c f|_{\{0\}} \geq T_k \wedge dd^c \Psi_{f,0}|_{\{0\}},$$

that gives us

Proposition 6 *Let $f \in PSH_-(\Omega)$ be locally bounded out of $\{0\}$ and $\exp f \in C(\Omega)$. Then*

$$\begin{aligned} (dd^c f)^n|_{\{0\}} &\geq (dd^c f)^{n-1} \wedge dd^c \Psi_{f,0}|_{\{0\}} \geq \dots \\ &\geq (dd^c f)^{n-k} \wedge (dd^c \Psi_{f,0})^k|_{\{0\}} \geq \dots \geq (dd^c \Psi_{f,0})^n. \end{aligned}$$

4 Dirichlet problem with local indicators

Let Ω be a bounded pseudoconvex domain in \mathbf{C}^n and K be a compact subset of Ω . By $PSH(\Omega, K)$ we denote the class of plurisubharmonic functions on Ω that are locally bounded on $\Omega \setminus K$.

Let $K = \{x^1, \dots, x^N\} \subset \Omega$ and $\{\Psi_m\}$ be N indicators, i.e. circled functions in $PSH_-(D)$ whose convex images satisfy (18). In the sequel we assume that $\Psi_m \in PSH(D, \{0\})$. Then by Proposition 4,

$$(dd^c \Psi_m)^n = \tau_m \delta(0), \quad 1 \leq m \leq N. \quad (30)$$

Let us fix the system $\Phi = \{(x^1, \Psi_1), \dots, (x^N, \Psi_N)\}$ and consider a positive measure T_Φ on Ω , defined as

$$T_\Phi = \sum_{1 \leq m \leq N} \tau_m \delta(x^m). \quad (31)$$

Each function Ψ_m can be extended from a neighbourhood of the origin to a function $\tilde{\Psi}_m \in PSH(\mathbf{C}^n, \{0\})$, and the indicators of the functions

$$\tilde{\Psi}(x) = \sum_m \tilde{\Psi}_m(x - x^m) + A, \quad (32)$$

at x^m are equal to Ψ_m , $1 \leq m \leq N$, for any real number A . So the class

$$N_{\Phi, \Omega} = \{v \in PSH_-(\Omega, K) : \Psi_{v, x^m} \leq \Psi_m, \quad 1 \leq m \leq N\} \quad (33)$$

is not empty.

Theorem 1 implies

Theorem 2 $(dd^c f)^n \geq T_\Phi \quad \forall f \in N_{\Phi, \Omega}.$

Now we introduce the function

$$G_{\Phi, \Omega}(x) = \sup\{v(x) : v \in N_{\Phi, \Omega}\}. \quad (34)$$

Theorem 3 *Let Ω be a hyperconvex domain in \mathbf{C}^n . Then the function $G = G_{\Phi, \Omega}$ has the following properties:*

- a) $G \in PSH_-(\Omega, K);$
- b) $G(x) \rightarrow 0$ as $x \rightarrow \partial\Omega;$
- c) $\Psi_{G, x^m} = \Psi_m, \quad 1 \leq m \leq N;$
- d) $(dd^c G)^n = T_\Phi$, the measure T_Φ being defined by (31);
- e) $G \in C(\overline{\Omega} \setminus K).$

Remark. In the case where $\Psi_m = \nu_m \log |x|$, the function $G_{\Phi, \Omega}$, the pluricomplex Green function with several weighted poles, was introduced in [16]. A situation with infinite number of poles was considered in [19], where a function $G_{f, \Omega}$ was introduced as the upper envelope of the class $\{v \in PSH_-(\Omega, K) : \nu(dd^c v, x) \geq \nu(dd^c f, x), \forall x\}$, f being a plurisubharmonic function with the following properties: e^f is continuous, $f^{-1}(-\infty)$ is a compact subset of Ω , and the set $\{x : \nu(dd^c f, x) > 0\}$ is dense in $f^{-1}(-\infty)$. Our proof is much the same as of the corresponding statements of [19].

Proof of Theorem 3. Since $N_{\Phi, \Omega} \neq \emptyset$, the function $G = G_{\Phi, \Omega}$ is well defined and

$$G^* = \limsup_{y \rightarrow x} G(y) \in PSH_-(\Omega, K).$$

The function $\tilde{\Psi}$ in (32) can be modified in a standard way to $\tilde{\Psi}' \in PSH_-(\Omega, K)$ such that $\tilde{\Psi}'(x) = \alpha \rho(x)$ in a neighbourhood of $\partial\Omega$, α being a positive number and ρ a bounded exhaustion function on Ω , and $\tilde{\Psi}'(x) = \tilde{\Psi}(x) - \beta$ on a neighbourhood of K . It shows us that

$$G^* \geq \tilde{\Psi}'. \quad (35)$$

It implies, in particular, that

$$\Psi_{G^*, x^m} \geq \Psi_{\tilde{\Psi}', x^m} = \Psi_m, \quad 1 \leq m \leq N. \quad (36)$$

Since $\Psi_{\sup\{v,w\},x} \leq \sup\{\Psi_{v,x}, \Psi_{w,x}\}$ for any plurisubharmonic functions v and w , there exists an increasing sequence of functions $v_j \in N_{\Phi,\Omega}$ such that $\lim_{j \rightarrow \infty} v_j = v \leq G$ and $v^* = G^*$.

The indicator of v_j at x^m is the limit of $R^{-1}g_{v_j,x^m}(R \log |y_k|)$ for $R \rightarrow +\infty$, the function $g_{v_j,x^m}(u)$ being the convex image of the mean value of $v_j(x_k^m + e^{u_k + i\theta_k})$ with respect to θ_k for $u_k < \log \text{dist}(x^m, \partial\Omega)$, $1 \leq k \leq n$, and the limit is obtained by the increasing values. It gives us

$$R^{-1}g_{v_j,x^m}(R \log |y_k|) \leq \Psi_{v_j,x^m}(y) \leq \Psi_m(y). \quad (37)$$

The functions v_j increase to G^* out of a pluripolar set $X = \{x \in \Omega : v(x) < v^*(x)\}$. Since the restriction of X to the distinguished boundary of any polydisk is of zero Lebesgue measure [14], (37) implies that

$$R^{-1}g_{G^*,x^m}(R \log |y_k|) \leq \Psi_m(y).$$

and thus, taking $R \rightarrow +\infty$,

$$\Psi_{G^*,x^m} \leq \Psi_m, \quad 1 \leq m \leq N. \quad (38)$$

As $G^* \in PSH_-(\Omega)$, the function G^* belongs to the class $N_{\Phi,\Omega}$ and so

$$G^* \equiv G. \quad (39)$$

By (36) and (38), $\Psi_{G,x^m} = \Psi_m$. It proves statements *a*) and *c*); statement *b*) follows from inequality (35).

Continuity of G can be proved as in [19], Theorem 2.6, with the following modification. Instead of Demailly's approximation theorem [8] we use the similar fact: for any function $u \in PSH(\Omega)$ there exists a sequence of continuous plurisubharmonic functions u_m satisfying

$$u(x) - \frac{c_1}{m} \leq u_m(x) \leq \sup \{u(x+y) : |y_k - x_k| \leq r_k, 1 \leq k \leq n\} + \frac{1}{m} \log \frac{c_2}{r_1 \dots r_n}$$

and

$$\Psi_{u,x}(y) \leq \Psi_{u_m,x}(y) \leq \Psi_{u,x}(y) - \frac{1}{m} \log |y_1 \dots y_n|, \quad \forall x \in \Omega, \forall y \in D.$$

To prove *d*), observe that in view of (35) and (39), $\tilde{\Psi}' \leq G$. By the comparison theorem of Demailly ([7], Theorem 5.9), this implies

$$(dd^c G)^n|_{\{x^m\}} \leq (dd^c \tilde{\Psi}')^n|_{\{x^m\}} \leq [dd^c \Psi_m(x - x^m)]^n, \quad 1 \leq m \leq N,$$

and therefore

$$(dd^c G)^n|_K \leq T_\Phi. \quad (40)$$

On the other hand, by Theorem 2,

$$(dd^c G)^n \geq T_\Phi.$$

Being comparing to (40) this provides

$$(dd^c G)^n|_K = T_\Phi.$$

Finally, the equality $(dd^c G)^n = 0$ on $\Omega \setminus K$ can be proved in a standard way by showing it is maximal on $\Omega \setminus K$ (see [1], [6]), that proves d).

The theorem is proved.

As a consequence, we get an "indicator" variant of the Schwarz type lemma (see [16], [19]):

Theorem 4 *Let the indicator of a function $g \in PSH(\Omega)$ at x^m does not exceed Ψ_m , $1 \leq m \leq N$, and let $g(x) \leq M$ on Ω . Then $g(x) \leq M + G_{\Phi, \Omega}(x)$, $\forall x \in \Omega$.*

Now we are going to show that the function $G_{\Phi, \Omega}$ is the unique plurisubharmonic function with the properties $a) - d)$ of Theorem 3. It is known that for unbounded plurisubharmonic functions u , the Dirichlet problem

$$\begin{cases} (dd^c v)^n = \mu \geq 0 & \text{on } \Omega \\ v = h & \text{on } \partial\Omega \end{cases} \quad (41)$$

need not have a unique solution even in a simple case $\mu = \delta(0)$, $h \equiv 0$. However, a solution is unique under some regularity assumptions on the functions v . For example, as was established in [19], (41) has a unique solution for

$$\mu = \sum [\nu(dd^c f, x^m)]^n \delta(x^m) \quad (42)$$

with $f(x)$ specified in the remark after the statement of Theorem 3, if the functions $v(x) \in PSH_-(\Omega, K)$ have to satisfy

$$\nu(dd^c v, x^m) = \nu(dd^c f, x^m), \quad 1 \leq m \leq N. \quad (43)$$

These additional relations mean that

$$v(x) \sim \nu(dd^c v, x^m) \log |x - x^m| \text{ near } x^m \quad (44)$$

(v has regular densities at its poles, in the terminology of [16]).

In our situation,

$$\mu = T_f = \sum_m \tau_m \delta(x^m), \quad (45)$$

where τ_m are defined by (30) with $\Psi_m = \Psi_{f, x^m}$, and we are going to replace condition (43) by $\Psi_{v, x^m} = \Psi_{f, x^m}$, $1 \leq m \leq N$.

To prove the uniqueness, we need a variant of the comparison theorem for unbounded plurisubharmonic functions (see [1] - [4], [6], [11], [19] for different classes of plurisubharmonic functions).

Theorem 5 *Let $f \in PSH(\Omega, K)$, $K = \{x^1, \dots, x^m\}$, and*

$$(dd^c f)^n|_K = T_f, \quad (46)$$

the measure T_f being given by (45). Let $v \in PSH(\Omega, K)$ satisfy the conditions

- 1) $\liminf_{x \rightarrow \partial\Omega} (f(x) - v(x)) \geq 0$;
- 2) $(dd^c v)^n \geq (dd^c f)^n$ on $\Omega \setminus K$;
- 3) $\Psi_{v, x^m} \leq \Psi_{f, x^m}$, $1 \leq m \leq N$.

Then $v \leq f$ on Ω .

The proof is just as of Theorem 3.3 of [19], and we omit it here.

Corollary 1 *Under the conditions of Theorem 3, the function $G_{\Phi, \Omega}$ is the unique plurisubharmonic function with the properties a) – d) of that theorem.*

Remarks.

1. Condition (46) is essential. Indeed, let

$$f(x) = \frac{1}{2} \log(|x_1|^4 + |x_1 + x_2^2|^2), \quad v(x) = \frac{1}{2} \log(|x_1|^2 + |x_2|^4) + m$$

with $m = \inf \{f(x) : |x| = 1\} > -\infty$. Then $v(x) \leq f(x)$ for $|x| = 1$, $(dd^c f)^2 = (dd^c v)^2 = 0$ on $\{0 < |x| < 1\}$, and $\Psi_{v, 0}(x) = \Psi_{f, 0}(x) = \log \max \{|x_1|, |x_2|^2\}$. However, for $x_2 = t \in (0, e^m)$, $x_1 = -x_2^2$, $f(x) = 2 \log t < \log t + m < v(x)$. The reason here is that $(dd^c f)^2 = 4\delta(0) > 2\delta(0) = T_f$.

2. By Comparison theorem of Demailly [7], relation (46) is true when

$$f(x) \sim \Psi_{f, x^m}(x - x^m) \text{ near } x^m,$$

a weaker than (44) but still controlled regularity.

3. By Proposition 5, an indicator Ψ possesses the properties a) – d) of $G_{\Phi, \Omega}$ from Theorem 3 with $\Omega = D$, the unit polydisk, and $\Phi = (\{0\}, \Psi)$. Therefore, $\Psi = G_{D, \Phi}$.

Theorems 3 and 5 allow us also to state the following result.

Theorem 6 *Let Ω be a bounded strictly pseudoconvex domain, $K = \{x^1, \dots, x^m\}$, and let a function $f \in PSH(\Omega, K)$ satisfy*

$$(dd^c f)^n = T_f.$$

Then the Dirichlet problem

$$\begin{cases} (dd^c v)^n = T_f & \text{on } \Omega \\ \Psi_{v,x^m} = \Psi_{f,x^m} & \text{for } 1 \leq m \leq N \\ v = h & \text{on } \partial\Omega \end{cases}$$

has a unique solution in the class $PSH(\Omega, K)$ for each function $h \in C(\partial\Omega)$. This solution is continuous on $\overline{\Omega} \setminus K$.

Proof. Let $\Phi = \{(x^m, \Psi_m)\}$ with $\Psi_m = \Psi_{f,x^m}$. Consider the class

$$N_{f,h} = \{v \in PSH(\Omega, K) : \Psi_{v,x^m} \leq \Psi_{f,x^m} \ \forall m, \lim_{x \rightarrow y} v(x) = h(y) \ \forall y \in \partial\Omega\}.$$

Let $u_0(x)$ be the unique solution of the corresponding homogeneous problem

$$\begin{cases} (dd^c u)^n = 0 & \text{on } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Then $u_0 + G_{\Phi,\Omega} \in N_{f,h}$, so $N_{f,h} \neq \emptyset$.

The desired solution v_0 is given as

$$v_0(x) = \sup \{v(x) : v \in N_{f,h}\}.$$

Just as in the proof of Theorem 3, one can show that v_0 does solve the problem and is continuous on $\overline{\Omega} \setminus K$. The uniqueness follows from Theorem 5.

Theorem 6 can be related to the following question which was one of the motivations of the present study. Let $F : \overline{\Omega} \rightarrow \mathbf{C}^n$ be a holomorphic mapping with isolated zeros $\{x^m\} \subset \Omega$ of multiplicities μ_m . Then the function $f(x) = \log |F(x)|$ solves the Dirichlet problem

$$\begin{cases} (dd^c v)^n = \sum_m \mu_m \delta(x^m) & \text{on } \Omega \\ v = f & \text{on } \partial\Omega. \end{cases}$$

Under what extra conditions on v , the function f is the unique solution of the problem? By Theorem 6, if f has regular behaviour at x^m with respect to its indicators, i.e. if

$$(dd^c \Psi_{f,x^m})^n = \mu_m \delta(x^m), \ 1 \leq m \leq N,$$

it gives the unique solution to the problem

$$\begin{cases} (dd^c v)^n = \sum \mu_m \delta(x^m) & \text{on } \Omega \\ \Psi_{v,x^m} = \Psi_{f,x^m} & 1 \leq m \leq N \\ v = f & \text{on } \partial\Omega. \end{cases}$$

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